obtained from (2) by replacing the original d and a by D and A the corresponding constants for any other circle, imposing the restriction $A^2-D^2=n^2$, and introducing K as the longitude of the center of the circle. There is a double infinitude of these circles, obtained by varying D and K.

Moreover, as will now be shown, there exist real initial conditions for which any one of these circles is described as a central orbit.*

4. CIRCULAR ORBITS THROUGH A POINT. *Theorem I. Through any point* (u_1, θ_1) *there passes in each direction one and only one circle of the family* (5). For, if the given direction makes with the radius vector to that point an angle ψ_1, then, since $\tan\psi=r\,d\theta/dr=-u\,d\theta/du$, equation (3) gives for any circle of (5) on replacing a by A and a^2-d^2 by n^2:

$$u_1^2\csc^2\psi_1=4A^2u_1^4[n^4u_1^2+1]^{-2}. \tag{6}$$

This equation determines a single real positive value of A, which is never less than n, since $\csc^2\psi_1\geq 1$ and $n^2u_1^2+1\geq 2nu_1$:

$$A^2=\csc^2\psi_1.(n^2u_1^2+1)^2/4u_1^2\geq n^2. \tag{7}$$

Hence $D^2=A^2-n^2$ gives a real value for D. The substitution of the latter value, with the coordinates of (u_1, θ_1) in (5) determines K uniquely (except for additive multiples of 2π) and real.

Theorem II. There is a unique velocity with which a particle projected at any point in any direction will describe a circle. For, by the preceding theorem, A is determined uniquely; and hence $HA=m$ determines H uniquely. And since $H=v_1r_1\sin\psi_1$ in any central orbit, there is but one possible value for v_1. Now to describe the required circle with the determined value of H, the particle would have to be attracted by a force whose law is precisely (4); hence, conversely, projection in the given direction with the velocity v_1 will result, under the law (4), in a circular orbit.

5. PROPERTIES OF THE FAMILY OF ORBITS. From the constancy of AH it follows that the constant of areas required in any circular orbit decreases as the radius of the circles increases. From the constancy of A^2-D^2 it follows that the radius of the circle must always increase with the distance of its center from the pole. There is no upper limit for A or D, but there is a lower limit (not zero) for A: viz., the smallest possible circular orbit is that one whose center lies at the center of force. But the two invariants have more exact interpretations, which are both simple and interesting.

*In a paper "On the Law of Gravitation in the Binary Systems," [*American Journal of Mathematics*, vol. 31, pp. 62-85], the writer discussed the family of conics by a law of which Type I is a limiting case. The present family of circles, however, is not included among those conics; for each law there considered admitted only one circular orbit, and the center was at the pole. Certain questions raised and answered below would not have arisen in connection with the family of conics.

Property I. The curvature at either apse is proportional to the constant of areas required for the orbit. For the curvature is $1/A$ which equals H/m.

Property II. The product of the two apsidal distances is constant. For these distances are $A-D$ and $A+D$ whose product equals n^2.

Property III. All circular orbits whose apsidal radii vectores are collinear pass through two fixed points on the perpendicular drawn to the line of centers at the center of force. For the half-chord intercepted on this perpendicular, being a mean proportional between $A-D$ and $A+D$, is constant and equal to n

Another (and more useful) statement of this property is that *the vectorial angle from either apse to the radius vector of length n is constant and equal to* $\pi/2$.

§ 3. Other Laws Admitting Properties I—III.

6. Admitting Property I. Two of these three properties were found also in the family of conics mentioned above; and it is natural to inquire whether there are still other laws of force whose trajectories will possess such properties. Attention will be confined to those cases where the force is a continuous function of the distance.

Let the force be denoted by $u^2P(u)$, and let $F(u)=2\int P(u)du$. [Since the continuity of $P(u)$ follows from the hypothesis that f is a continuous function of r, $P(u)$ is integrable; thus $F(u)$ exists and is continuous.]

A first observation is that any such force, whether attractive or repulsive, admits families of orbits having pericenters at arbitrary distances; and, if everywhere attractive, admits also families having apocenters at arbitrary distances. For a particle may be so projected at any distance $1/\beta$ that the initial direction is perpendicular to the initial radius vector, so that $du/d\theta=0$ for $u=\beta$; and the initial velocity may be such as to give h any desired value. Moreover the standard differential equation of central orbits,

$$f=h^2u^2(u+d^2u/d\theta^2)=u^2P(u), \tag{8}$$

shows that $d^2u/d\theta^2 \gtrless 0$ at $u=\beta$ according as $h^2 \gtrless P(\beta)/\beta$. Hence whatever the sign of $P(\beta)$, h may be chosen so large that $d^2u/d\theta^2<0$, making u a maximum at the initial point. And, if $P(\beta)>0$ [force attractive], it is also possible to choose h so small that u is a minimum at $u=\beta$.

Secondly, since the expression for the curvature [see No. 2] becomes at an apse $u+d^2u/d\theta^2$, or simply $P(u)/h^2$, Property I would require: $h=\lambda P(\beta)/h^2$, or

$$h^3=\lambda.P(\beta), \tag{9}$$

where λ is a constant factor of proportionality. Thus for *any* value of λ, h is defined as a function of β so as to insure Property I. It remains to be shown simply that λ may have such values that h, defined by (9), will meet the condition above for the existence of a pericenter or apocenter at $u=\beta$. Now to have $h^2>P(\beta)/\beta$ is to have $[\lambda P(\beta)]^2 \gtrless [P(\beta)/\beta]^3$, or $\lambda^2 \gtrless P(\beta)/\beta^3$. But for any interval of β, $0<A \leqq \beta \leqq B$, $P(\beta)$ being continuous has an upper bound M. Hence, if λ^2 be chosen greater than M/A^3, the condition is satisfied. And if $P(\beta)>0$ always, the reverse inequality may be treated in like manner.

Therefore, *every central force which is a continuous function of the distance admits families of orbits having pericenters and possessing Property I; and, if everywhere attractive, families of orbits with apocenters, having that property.*

Forces admitting orbits with both a pericenter and an apocenter are somewhat special as will appear in the next paragraph. For the orbits to have Property I at their pericenters only, any continuous force whose trajectories have two apses is admissible, the corresponding apocenters being determined as follows: One integration of (8) gives

$$h^2(u^2+(du/d\theta)^2)=c+F(u), \tag{10}$$

where c is an arbitrary constant. If β and α denote, respectively, the pericentral and apocentral values of u, then

$$h^2(\beta^2-\alpha^2)=F(\beta)-F(\alpha). \tag{11}$$

Now if β be allowed to vary, and h be determined by (9), then (11) will determine α in those cases where the orbit has two apses. If, however, Property I is to hold for both apses, there is a further restriction upon the force, given by $\lambda P(\beta)=\mu P(\alpha)$, μ being the factor of proportionality for apocenters. A case of this sort is treated later [in No. 9].

7. ADMITTING PROPERTY II. If an orbit has two apsidal distances, $1/\alpha$ and $1/\beta$, Property II requires simply that $\alpha\beta=k^2$, a constant. As remarked, however, there is a second category of forces none of whose trajectories have two apsidal distances; an example is the case where the force varies inversely as the cube of the distance. Still other laws admit trajectories consisting of two branches separated by a region of imaginary motion, —one branch having only a pericenter and extending to infinity, the other having only an apocenter and reaching to the center of force. In such a case which branch is actually described in the motion of the particle depends upon the initial distance of the particle; and the path really has but one apse. It is easy to find conditions upon the force, which are necessary and sufficient to place it in any of the three categories.

Theorem I. In order that a force, f, shall admit orbits extending from one apse to another, it is necessary and sufficient that f/u^3 be a decreasing function of u throughout some interval of u, say $[a, b]$. For if there is an apocenter at $u=\alpha$ and a pericenter at $u=\beta$ $[\beta>\alpha]$, the necessary constant of areas is given by (11); and moreover it is necessary that $h^2 d^2u/d\theta^2$, or $-h^2u+P(u)$, be positive for $u=\alpha$ and negative for $u=\beta$; thus

$$P(\alpha)/\alpha > h^2 > P(\beta)/\beta. \tag{12}$$

If $P(u)/u$ is nowhere a decreasing function of u, it is clear that the inequality (12) is impossible; thus the hypothesis is necessary.

It is also sufficient; for let $\alpha=a$ and $\beta=b$. Then the use of the generalized theorem of mean value in (11) gives the constant of areas for which $du/d\theta$ vanishes at $u=\alpha$ and at $u=\beta$:

$$h^2 = \frac{F(\beta)-F(\alpha)}{\beta^2-\alpha^2} = \frac{P(u_1)}{u_1}, \tag{13}$$

where $\alpha<u_1<\beta$, the theorem being valid here, since $F(u)$ and u^2 together with their derivatives are continuous functions in $[\alpha, \beta]$, and the latter dedominator does not vanish in the interval. Now by hypothesis,

$$\frac{P(\alpha)}{\alpha} > \frac{P(u_1)}{u_1} > \frac{P(\beta)}{\beta}; \tag{14}$$

so that (12) is satisfied, and there is an apocenter where $u=\alpha$ and a pericenter where $u=\beta$. Whether the orbit extends from $u=\alpha$ to $u=\beta$ or whether there are still other apses between these values of u (for the same values of h) does not affect the theorem. For consider the next apsidal value above $u=\alpha$, say $u=\alpha'$. Then since $h^2=P(u_2)/u_2$ where $\alpha<u_2<\alpha'$; and since this shows that $h^2>P(\alpha')/\alpha'$, the apse at $u=\alpha'$ is a pericenter. A similar argument shows that the apse nearest to $u=\beta$, say $u=\beta'<\beta$, would have to be an apocenter. In any event the same branch has both a pericenter and an apocenter.

Corollary 1. If f/u^3 decreases everywhere, then any two distances can be the apsidal distances in an orbit. For the constant of areas required for the vanishing of $du/d\theta$ would be such as to give a pericenter at the lesser distance and an apocenter at the greater.

Theorem II. In order that a force, f, admit trajectories consisting of two branches, one having an apocenter only, and one a pericenter only, it is sufficient that f/u^3 be everywhere an increasing function of u, and necessary that f/u^3 be an increasing function in some interval $[a, b]$. For the existence of a pericenter at $u=\beta$ and an apocenter at $u=\alpha$ $[\alpha>\beta]$ requires the

inequality (12), which can be satisfied only if f/u^3 is an increasing function in some interval between β and α. Further, since (14) is satisfied if f/u^3 is everywhere an increasing function, the apses at $u=\alpha$ and $u=\beta$ must be, respectively, an apocenter and a pericenter; and there are no further apses by Theorem I, the necessary condition not being fulfilled.

Corollary 2. If f/u^3 is constant [the case of the inverse cube of the distance], *only one apse is possible in a trajectory.* For the conditions necessary for two apses either in the same branch or in different branches are not satisfied.

Corollary 3. There is no apsidal distance between $1/\alpha$ and $1/\beta$ in the orbits of Theorem I. For between the apocentral value $u=\beta'$ and the pericentral value $u=\alpha'$, f/u^3 would have to be somewhere an increasing function.

Theorem III. Any law admitting trajectories with two apsidal distances admits families having Property II.

Case I. When there is an interval $[a, b]$ throughout which f/u^3 decreases, any two apsidal values α and β $[a \leqq \alpha < \beta \leqq b]$ determine a value of h for which the orbit lies in the region $u=a$ to $u=b$ and has two apses. Let $ab=k^2$, and choose arbitrarily $k \leqq \alpha \leqq a$, and $\beta=k^2/\alpha$. Then the orbit lies between $u=\alpha$ and $u=\beta$, and has Property II.

Case II. When f/u^3 always increases with u, select k and α arbitrarily $[\alpha > k]$, and let $\beta=k^2/\alpha$. Then the vanishing of $du/d\theta$ for $u=\alpha, \beta$, determines a value of h, $h^2=P(u_1)/u_1$, such that $P(\alpha)/\alpha < h^2 < P(\beta)/\beta$. Thus the trajectory has two branches of the kind considered in Theorem II. By varying α a family is obtained having Property II.

8. ADMITTING PROPERTY III. Not all forces admit orbits in which the angle from the radius vector of a given length to the nearest pericenter is $\pi/2$. For example, the laws represented by $f=kr^n$ where $n>1$ have their apsidal angles all less than $\pi/2$.* But for any force, f, such that $u.f$ is an increasing function of u, all apsidal angles are greater than $\pi/2$; and by Corollary 1, if also f/u^3 is everywhere a decreasing function, any pair of apsidal values α and β are admissible.

Let c be an arbitrarily chosen constant; and let θ denote the angle from the radius vector where $u=c$ to the nearest pericentral line, where $u=\beta>c$. Then

$$\theta=\int_c^\beta \frac{du}{\sqrt{\{\beta^2-u^2+[F(u)-F(\beta)]/h^2\}}}. \tag{15}$$

Now for sufficiently great values of h, this integral differs arbitrarily little from

*This and the following statement are established by a test contained in an unpublished paper by the writer, presented to the American Mathematical Society, December 30, 1908.

(16) $$\int_\beta^c du/\sqrt{(\beta^2-u^2)},\ \text{or}\ \pi/2-\arcsin(c/\beta);$$

hence there are values of h large enough to make $\theta<\pi/2$. But if α be chosen sufficiently near c [$\alpha<c$], $\Theta-\theta$ may be made arbitrarily small [Θ denoting the apsidal angle]; so that, since $\Theta>\pi/2$, h can be so chosen that $\theta>\pi/2$. Between this value of h and the "sufficiently great values," there must exist a value for which $\theta=\pi/2$.

By varying β a family of orbits is obtained having Property III. For any value of β the necessary value of h is given by (15) on placing $\theta=\pi/2$; and α is given by (11) when β and h are known.

There is likewise a family having Property III with respect to the apocenters; for with a given α, h can be selected so small that θ shall be arbitrarily small; or β selected so near c that θ shall be greater than $\pi/2$. Hence, for a certain value of h, $\theta=\pi/2$.

The determination of h in each case is unique; for in (15) if h be assigned two different values h_1 and h_2, $h_2>h_1$, the integrand involving h_2 is smaller throughout than that involving h_1, so that only for one of these values can $\theta=\pi/2$.

9. ADMITTING PROPERTIES I AND II. The preceding sections show that no one of the three properties serves to characterize any particular law of force. But it is quite otherwise if laws be sought admitting both Properties I and II.

For if the constant of areas be proportional to the curvature at each apse; then $h^3=\lambda P(\beta)=\mu P(\alpha)$, where λ and μ are constants. And if the family is to have $\alpha\beta=k^2$, λ and μ must be equal, since as α and β each approach the value k, h^3 approaches $\lambda P(k)$ or $\mu P(k)$. Therefore, also $P(k^2/\beta)=P(\beta)$; and from (11) follows

(17) $$F(\beta)-F(k^2/\beta)=(\beta^2-k^4/\beta^2)\,[\lambda.P(\beta)]^{2/3},$$

which must hold as β varies through the range of values taken in the family. Differentiation of (17) with respect to β gives on replacing $P(k^2/\beta)$ by $P(\beta)$:

(18) $$(1+k^2/\beta^2)P(\beta)=(\beta+k^4/\beta^3)\,[\lambda P(\beta)]^{2/3} + (\lambda/3)(\beta^2-k^4/\beta^2)\,[\lambda P(\beta)]^{-1/3}P'(\beta),$$

or substituting kx for β and $\phi(x)$ for $\lambda.P(kx)$:

(19) $$\frac{d\phi}{dx}+3\frac{x^4+1}{x^5-x}\phi=\frac{3}{\lambda k}\cdot\frac{\phi^{\frac{1}{3}}}{x^2-1},$$

a differential equation which is of Bernoulli's type, and is reducible to a lin-

ear equation by the substitution $\phi = y^{-3}$. A particular integral is $y = (x^2+1)/2\lambda kx$, while the solution of the auxiliary equation is $y = c\sqrt{(x^4-1)}/2\lambda kx$; so that the general solution is

$$\phi^{-\frac{1}{3}} = y = [c\sqrt{(x^4-1)} + x^2 + 1]/2\lambda kx, \tag{20}$$

where c is an arbitrary constant; or

$$P(\beta) = 8k^6\lambda^2\beta^3[c\sqrt{(\beta^4-k^4)} + \beta^2 + k^2]^{-3}. \tag{21}$$

Hence the most general law of force, $f = u^2P(u)$, would be

$$f = 8k^6\lambda^2 \cdot \frac{u^5}{[c\sqrt{(u^4-k^4)} + u^2 + k^2]^3}. \tag{22}$$

But, whether c is real or imaginary [$c \neq 0$], (22) gives imaginary values for f either when $u > k$ or else when $u < k$; and such a force would not admit real orbits in both parts of the plane. Thus the only admissible value of c is zero; and (22) then reduces to the same form as (4).

Hence *laws of Type I are the most general which admit families of orbits possessing Properties I and II;* and these two properties serve, therefore, to characterize laws admitting circular orbits whose centers are elsewhere than at the center of force.

§ 4. The Families for Particular Laws.

10. For Newton's Law. Every orbit described under a force operating according to Newton's law is a conic having a facus at the center of force. Expressed in terms of the constants of the conic, the force is given by

$$f = h^2u^3/a(1-e^2), \tag{23}$$

so that, for all the orbits, $h^2/a(1-e^2)$ must be constant. If also h varies as the curvature at any apse, $1/a(1-e^2)$ or a/b^2, it follows that $a(1-e^2)$, and hence also h, must be constant in the family. Hence, for the Newtonian law, the family having Property I is the *totality of conics* having a certain curvature at a vertex, or what is equivalent, *having latera recta of a certain length.* Evidently there is an infinitude of such families, obtained by changing the arbitrary constant. And further, each of these families possesses Property III also.

Again, to have the product of the apsidal distances constant requires the constancy of $a^2(1-e^2)$, or of the minor axis. Hence for Newton's law,

the family possessing Property II is the set of conics having minor axes of an arbitrary constant length.

11. FOR THE LAW OF THE DIRECT DISTANCE. All orbits described under a force varying directly as the distance, are conics whose centers lie at the center of force. The force, expressed in terms of the constants of the conics, is given by

$$f=h^2/a^2b^2u; \tag{24}$$

so that h^2/a^2b^2 must be constant for all the orbits. If also h is proportional to a/b^2, it is necessary that b^6, and hence also b, remain invariant. Thus, for this law, the family possessing Property I is the set of conics having minor axes of a given length. The constant of areas must vary as the major axis in this family. To have Property I at the pericenters, interchange b and a in the preceding statements. Each of these families has Property III also.

Finally, to have the product of the apsidal distances constant requires the constancy of $a.b$ and also of h. Hence, for the law of the direct distance, the family possessing Property II is the set of conics having the product of the axes constant, or what is equivalent for ellipses, having a given (arbitrary) area.

ON THE IRREDUCIBILITY OF CERTAIN POLYNOMIALS.

By JACOB WESTLUND, Purdue University.

The object of the following note is to determine whether the two polynomials

$$f_1(x)=(x-a_1)(x-a_2)\ \ldots\ (x-a_n)-1 \text{ and}$$
$$f_2(x)=(x-a_1)(x-a_2)\ \ldots\ (x-a_n)+1,$$

where $a_1, a_2, \ldots, a_n$ are distinct integers, are reducible or irreducible.

Let us first consider $f_1(x)$. If $f_1(x)$ were reducible we would have

$$f_1(x)=\phi(x)\psi(x),$$

where $\phi(x)$ is irreducible and of a lower degree than n. Then since

$$\phi(a_i)\psi(a_i)=-1,\ i=1, 2, \ldots, n,$$

we must have

$$\phi(a_i)=\pm 1 \text{ and } \psi(a_i)=\mp 1.$$

Hence,

$$\phi(a_i)+\psi(a_i)=0,\ i=1,\ 2,\ \ldots,\ n;$$

and hence the equation

$$\phi(x)+\psi(x)=0,$$

whose degree is less than n, has n distinct roots, which is impossible. Hence, $f_1(x)$ is always irreducible.

Let us next consider $f_2(x)$. If $f_2(x)$ were reducible, we would have

$$f_2(x)=\phi(x)\psi(x),$$

where $\phi(x)$ is irreducible and of a lower degree than n. Then reasoning in the same way as in the first case we find that the equation

$$\phi(x)-\psi(x)=0,$$

which is of a lower degree than n, has n distinct roots. But this is impossible, unless $\phi(x)$ and $\psi(x)$ are identically equal. Hence the only case when $f_2(x)$ is reducible is when it is a perfect square, in which case n of course must be even.

A METHOD OF COMPUTING LOGARITHMS.

By C. E. WHITE, Nashville, Tennessee.

From the expansion $f(x)=f(a)+(x-a)f'(a)+\frac{(x-a)^2}{2!}f''(a)+\ldots.$ we derive by letting $a=x+h$ and $a=x-h$,

$$f(x)=f(x+h)-hf'(x+h)+\frac{h^2}{2!}f''(x+h)-\frac{h^3}{3!}f'''(x+h)+\ldots$$

$$f(x)=f(x-h)+hf'(x-h)+\frac{h^2}{2!}f''(x-h)+\frac{h^3}{3!}f'''(x-h)+\ldots$$

The above expansions may be used to an advantage in computing log-

arithms if h be determined so that $x+h$ and $x-h$ are each equal to some number whose logarithm is known.

Let us choose h in the first series so that $x+h=1000$ and h in the second series so that $x-h=1000$.

$$f(x+h)=\log 1000,\ f'(x+h)=\tfrac{1}{1000}=.001,\ \frac{f''(x+h)}{2!}=.0000005.$$

$$\therefore \log x=\log 1000-(.001h+.0000005h^2+.000000000333h^3+\ldots) \text{ or}$$
$$\log_{10} x=3-M(.001h+.0000005h^2+.000000000333h^3+\ldots)$$

For the second series we get

$$\log_{10} x=3+M(.001h-.0000005h^2+.000000000333h^3+\ldots)$$
$$\log 999=3-.43429\times.001-.43429\times.0000005=2.99956549$$
$$\log 1001=3+.43429\times.001-.43429\times.0000005=3.00043407$$
$$\log 997=3-.43429(.003+.0000045)=2.99869516$$
$$\log 1003=3+.43429(.003-.0000045)=3.00130093$$

The advantage of this method in computing a table of logarithms is that two logarithms can be computed with little more work than is required to compute one.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

A very good solution of 309 was received from J. M. Arnold.

210. Proposed by B. F. FINKEL, Ph. D., Drury College, Springfield, Mo.

Simplify, $\log[\sqrt[3]{(137)}\sqrt[7]{(56)}\div\sqrt[10]{(187)}\sqrt[4]{(75)}]$.

Remark by the PROPOSER.

This problem presents no difficulty whatsoever. It was proposed with a view of ascertaining whether different results would be obtained. Only two solutions have been received. We should like to have a large number of solutions.

311. Proposed by S. G. BARTON, Ph. D., Clarkson School of Technology, Potsdam, N. Y.

Find, by Cardan's Method, the real root (4) of $x^3-6x^2+10x=8$.

Solution by G. I. HOPKINS, Instructor in Mathematics and Astronomy, High School, Manchester, N. H.

Substitute $y+2$ for x. Then $y^3-2y-4=0$.

Substitute $v+z$ for y. Then, from the well known formula,

$$v=\sqrt[3]{[2+\tfrac{10}{9}\sqrt{3}]} \text{ and } z=\sqrt[3]{[2-\tfrac{10}{9}\sqrt{3}]}.$$

$\therefore 2+\frac{10}{9}\sqrt{3}$ must be the cube of a binomial, the first term of which is 1, and the second term contains $\sqrt{3}$.

Assume $(1+a\sqrt{3})^3=2+\frac{10}{9}\sqrt{3}$; whence $1+3a\sqrt{3}+9a^2+3a^3\sqrt{3}=2+\frac{10}{9}\sqrt{3}$, or $3a(1+a^2)\sqrt{3}+9a^2=1+\frac{10}{9}\sqrt{3}$.

$\therefore 9a^2=1$; whence $a=\frac{1}{3}$.

$\therefore 1+\frac{1}{3}\sqrt{3}$ is the cube root of $(2+\frac{10}{9}\sqrt{3})$.

$\therefore v=1+\frac{1}{3}\sqrt{3}$ and $z=1-\frac{1}{3}\sqrt{3}$; $v+z=2=y$; and, therefore, $x=y+2=4$.

Also by quadratics as follows:

Multiplying by x, $x^4-6x^3+10x^2-8x=0$.

$$x^4-6x^3+9x^2+x^2-8x=0,$$
$$(x^2-3x)^2+x^2-3x-5x=0,$$
$$(x^2-3x)^2+(x^2-3x)=5x.$$

Adding x^2-3x, $(x^2-3x)^2+2(x^2-3x)=x^2+2x$.

Adding 1, $(x^2-3x)^2+2(x^2-3x)+1=x^2+2x+1$.

$\therefore x^2-3x+1=\pm(x+1)$ and $x=4$.

Also solved by J. Scheffer, G. B. M. Zerr, G. W. Hartwell, and a student in Olivet College.

CALCULUS.

268. Proposed by PROFESSOR R. D. CARMICHAEL, Anniston, Ala.

Determine $\phi(y)$, independent of u, so that the equation $\int_0^u (u-y)^{(2p-1)/2}\phi(y)\,dy=u^m$ is satisfied, p and m being positive integers and $m>p$. Do you notice properties of special interest for any special cases?

Solution by the PROPOSER.

The problem may be generalized with small increase in the difficulty of solution; thus:

It is required to find $\phi(y)$ a function of y alone from the equation

$$(1) \qquad f(u)=\int_0^u h(u, y)u^a\,\phi(y)\,dy,$$

where $h(u, y)$ is a homogeneous function of u, y of degree zero such that it may be expressed as a finite or an infinite series,

$$h(u, y)=\beta_0+\beta_1\frac{y}{u}+\beta_2\frac{y^2}{u^2}+\beta_3\frac{y^3}{u^3}+\ldots,$$

and where $f(u)$ is a given function which may be written in the form

$$f(u)=c_1u^{a_1}+c_2u^{a_2}+c_3u^{a_3}+\ldots, \quad a\leqq a_1<a_2<a_3<\ldots;$$

a and the a's being positive or negative, entire or fractional, subject to the given relations.

First consider the special case

$$c_iu^{a_i}=\int_0^u(\beta_0+\beta_1\frac{y}{u}+\beta_2\frac{y^2}{u^2}+\ldots)u^a\,\phi_i(y)\,dy. \tag{2}$$

Now

$$\int_0^u(\beta_0+\beta_1\frac{y}{u}+\beta_2\frac{y^2}{u^2}+\ldots)\delta_iy^{a_i-a-1}dy$$

$$=\delta_iu^a\left(\frac{\beta_0}{a_i-a}y^{a_i-a}+\frac{\beta_1}{a_i-a+1}\frac{y^{a_i-a+1}}{u}+\frac{\beta_2}{a_i-a+2}\frac{y^{a_i-a+2}}{u^2}+\ldots.\right)\Bigg]_0^2$$

$$=\delta_i\,s_i\,u^{a_i}, \text{ where} \tag{3}$$

$$s_i=\frac{\beta_0}{a_i-a}+\frac{\beta_1}{a_i-a+1}+\frac{\beta_2}{a_i-a+2}+\frac{\beta_3}{a_i-a+3}+\ldots. \tag{4}$$

Equating the first member of (2) with the second of (3), we have

$$\delta_i=\frac{c_i}{s_i}. \quad \therefore\phi_i(y)=\frac{c_i}{s_i}y^{a_i-a+1}.$$

Evidently, then, a solution of (1) is

$$\phi(y)=\sum_i\phi_i(y); \text{ or } \phi(y)=\sum_i\frac{c_i}{s_i}x^{a_i-a-1},$$

where i runs over all the subscripts of c in the expression for $f(u)$ and s_i is defined by equation (4).

NOTE.—Volterra has studied a more general problem. See *Encyklop. d. Math. Wissensch.*, II, p, 808. The problem in the present form affords the solution of several problems in practical hydrodynamics.

NUMBER THEORY AND DIOPHANTINE ANALYSIS.

157. Proposed by A. H. HOLMES, Brunswick, Maine.

Find integral values for m and n in $64m^2n^2(m^2-n^2)^2+(m^2+n^2)^4=\square$.

No solution of this problem has been received.

158. Proposed by J. EDWARD SANDERS, Weather Bureau, Chicago, Ill.

Find positive rational values of a and b in the equation $x^4-2ax^2+x+a^2-b=0$, that will make each of the roots (all different) rational numbers.

Solution by E. B. ESCOTT, Ann Arbor, Mich.

Let two roots be $x=\alpha$ and $x=\beta$. Substituting, we have

$$b=\alpha^4-2a\,\alpha^2+\alpha+a^2=\beta^4-2a\,\beta^2+\beta+a^2.$$

Transposing, and removing factor $\alpha-\beta$ (since $\alpha\neq\beta$), we have $\alpha^3+\alpha^2\beta+\alpha\beta^2+\beta^3-2a(\alpha+\beta)+1=0$, *i. e.*,

$$a=\tfrac{1}{2}\left(\alpha^2+\beta^2+\frac{1}{\alpha+\beta}\right).$$

Substituting, we get $b=\frac{1}{4}\left(\alpha^2-\beta^2-\frac{1}{\alpha+\beta}\right)^2+\alpha$.

Substituting in the original equation, and removing the factors $x-\alpha$ and $x-\beta$, we have

$$x^2+(\alpha+\beta)x+\left(\alpha\beta-\frac{1}{\alpha+\beta}\right)=0.$$

This will have commensurable roots if

$$(\alpha+\beta)^2-4\left(\alpha\beta-\frac{1}{\alpha+\beta}\right)=(\alpha-\beta)^2+\frac{4}{\alpha+\beta}=r^2.$$

It is easily seen that α and β cannot be integral.

We can get as many rational values as we please by assuming any value for $\alpha+\beta$.

Example. Let $\alpha+\beta=\frac{1}{2}$. Then $\beta=\frac{1}{2}-\alpha$. Substituting in the last equation, we get $(2\alpha-\frac{1}{2})^2+8=r^2$. This can be satisfied in an infinite number of ways, *e. g.*,

$$2\alpha-\tfrac{1}{2}+r=4,\quad 2\alpha-\tfrac{1}{2}-r=-2,$$

whence $\alpha=\frac{3}{4}$, $\beta=-\frac{1}{4}$. Then $a=\frac{21}{16}$, $b=\frac{21}{16}$. The other roots are $\frac{5}{4}$ and $-\frac{7}{4}$.

Also solved by G. B. M. Zerr, and V. M. Spunar.

AVERAGE AND PROBABILITY.

192a. Proposed by A. H. HOLMES, Brunswick, Maine.

In a game of baccarat the dealer and each side of the table have two or three cards. The object is to get as near nine as possible, and tens and court cards do not count. If the first two cards dealt do not together amount to five, the player asks for another. If above five he does not. When the two cards amount to exactly five would the chances of the hand be bettered or diminished by drawing a third card, and how much?

Solution by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

Let A, B, C, D be the players, and, in order to avoid a multiplicity of solutions, we will assume cards for A, C, D when B has just five. Let A have an ace and a three; C, four and six; D, two and five, as follows:

A	C	D	B
1, 3	4, 6	2, 5	1, 4 or 2, 3 or 5, 10

In either case B betters his hand if he draws 1, 2, 3, 4, 5, 6, or 7; diminishes it if he draws 9 and leaves it the same if he draws 8, 10, or court cards. Since there are but 44 cards left in the pack, we have:

First
$\frac{2}{44}+\frac{3}{44}+\frac{3}{44}+\frac{2}{44}+\frac{3}{44}+\frac{3}{44}+\frac{4}{44}+\frac{4}{44}=\frac{10}{22}=\frac{5}{11}$=chance of bettering.
$\frac{4}{44}=\frac{1}{11}$=chance of diminishing.
$\frac{4}{44}+\frac{4}{44}+\frac{4}{44}+\frac{4}{44}+\frac{4}{44}=\frac{5}{11}$=chance of leaving same.

Second
$\frac{3}{44}+\frac{2}{44}+\frac{2}{44}+\frac{3}{44}+\frac{3}{44}+\frac{3}{44}+\frac{4}{44}=\frac{5}{11}$=chance of bettering.
The chance of diminishing or leaving the same is $\frac{1}{11}$ or $\frac{5}{11}$ as before.

Third
$\frac{3}{44}+\frac{3}{44}+\frac{3}{44}+\frac{3}{44}+\frac{2}{44}+\frac{3}{44}+\frac{4}{44}=\frac{21}{44}$=chance of bettering.
The chance of diminishing$=\frac{4}{44}=\frac{1}{11}$.
$\frac{4}{44}+\frac{4}{44}+\frac{4}{44}+\frac{4}{44}+\frac{3}{44}=\frac{19}{44}$=chance of remaining the same.

The solution is similar for A, C, D having other cards.

Hence we see that the drawing of a third card when the two cards count just five, is preferable. In the first and second cases, bettering to diminishing=5 : 1. In the third case, bettering to diminishing=21 : 4.

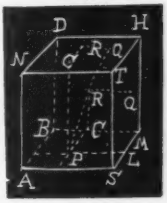
197. Proposed by HENRY HEATON, Belfield, N. D.

Solve No. 188 on the supposition that all lines having the same direction are equally distributed in space, and lines passing through the same point are distributed as the radii of a sphere drawn to points equally distributed.

Solution by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

We will solve (b) first.

Let AH be the cube, P the point of ingress, Q the point of egress of the hole. Let (x, y) be the coordinates of P. Through P, pass the plane RPC parallel to the face $ABDN$. Through Q draw the line QR parallel to BM and DH, where $BMDH$ is a face of the cube. Let $LM=x$, $LP=y$, $\angle RPC=\theta$, $\angle QRP=\phi$, $a=$ side of cube. When P, Q are in the adjacent faces $ASMB$ and $BMHD$, then $PR=x\sec\theta$, $PQ=x\sec\theta\sec\phi$. The limits of θ are 0 and $\tan^{-1}(a/x)=\theta_1$; of ϕ, 0 and $\tan^{-1}(y\cos\theta/x)=\phi_1$. When P is in $ASMB$ and Q in the opposite face $NTHD$, $PR=a\sec\theta$, $PQ=a\sec\theta\sec\phi$. The limits of θ are 0 and $\tan^{-1}(x/a)=\theta_2$; of ϕ, 0 and $\tan^{-1}(y\cos\theta/a)=\phi_2$; the limits of both x and y are 0 and a.



As P may be regarded as fixed, there are four faces adjacent to $ASMB$ and one opposite. The points of intersection of PQ with the surface of a sphere, center P and radius unity, is the required distribution. An element of surface of this sphere is $\cos\phi\, d\theta\, d\phi$. Hence the average length, L, is

$$L=\frac{\int_0^a\int_0^a\left[4\int_0^{\theta_1}\int_0^{\phi_1}x\sec\theta\sec\phi\cos\phi\, d\theta\, d\phi\right.}{\int_0^a\int_0^a\left[4\int_0^{\theta_1}\int_0^{\phi_1}\cos\phi\, d\theta\, d\phi\right.}\times$$

$$\times\frac{\left.+\int_0^{\theta_2}\int_0^{\phi_2}a\sec\theta\sec\phi\cos\phi\, d\theta\, d\phi\right]dx\, dy}{\left.+\int_0^{\theta_2}\int_0^{\phi_2}\cos\phi\, d\theta\, d\phi\right]dx\, dy}=\frac{N}{D}.$$

$$D=\int_0^a\int_0^a\left[4\int_0^{\theta_1}\frac{y\cos\theta}{\sqrt{(x^2+y^2\cos^2\theta)}}d\theta+\int_0^{\theta_2}\int\frac{y\cos\theta}{\sqrt{(a^2+y^2\cos^2\theta)}}d\theta\right]dx\, dy$$

$$=\int_0^a\int_0^a\left[4\sin^{-1}\left(\frac{ay}{\sqrt{(a^2+x^2)}\,\sqrt{(x^2+y^2)}}\right.\right.$$

$$\left.\left.+\sin^{-1}\left(\frac{xy}{\sqrt{(a^2+x^2)}\,\sqrt{(a^2+y^2)}}\right)\right]dx\, dy=\int_0^a\left[4a\sin^{-1}\left(\frac{a^2}{a^2+x^2}\right)\right.$$

$$\left.-4x\log\left[\frac{[a+\sqrt{(a^2+x^2)}]\,\sqrt{(a^2+x^2)}}{x[a+\sqrt{(2a^2+x^2)}]}\right]+a\sin^{-1}\left(\frac{x}{\sqrt{(2)}\,\sqrt{(a^2+x^2)}}\right)\right.$$

$$-x\log\left(\frac{\sqrt{(2)}\,[x+\sqrt{(a^2+x^2)}]}{x+\sqrt{(2a^2+x^2)}}\right)\Big]dx$$

$$=(a^2/12)\,[10\pi+24-45\sqrt{2}+21\sqrt{3}+6\log 2-24\log(1+\sqrt{3})+27\log(1+\sqrt{2})]$$
$$=2.6653a^2.$$

$$N=4\int_0^a\int_0^a\int_0^a \frac{x}{\sqrt{(x^2+z^2)}}\tan^{-1}\left(\frac{y}{\sqrt{(x^2+z^2)}}\right)dz\,dx\,dy$$

$$+\int_0^a\int_0^z\int_0^a \frac{a}{\sqrt{(a^2+z^2)}}\tan^{-1}\left(\frac{y}{\sqrt{(a^2+z^2)}}\right)dz\,dx\,dy$$

In the first term z is written for $x\tan\theta$, in the second term z is written for $a\tan\theta$.

$$N=4\int_0^a\int_0^a\left[\frac{ax}{\sqrt{(x^2+z^2)}}\tan^{-1}\left(\frac{a}{\sqrt{(x^2+z^2)}}\right)\right.$$

$$\left.-\tfrac{1}{2}x\log\left(\frac{a^2+x^2+z^2}{x^2+z^2}\right)\right]dz\,dx+\int_0^a\int_0^z\left[\frac{a^2}{\sqrt{(a^2+z^2)}}\tan^{-1}\left(\frac{a}{\sqrt{(a^2+z^2)}}\right)\right.$$

$$\left.-\tfrac{1}{2}a\log\left(\frac{2a^2+z^2}{a^2+z^2}\right)\right]dz\,dx=\int_0^a\left[4a\sqrt{(a^2+z^2)}\tan^{-1}\left(\frac{a}{\sqrt{(a^2+z^2)}}\right)\right.$$

$$\left.-4az\tan^{-1}(a/z)+2z^2\log\left(\frac{a^2+z^2}{z\sqrt{(2a^2+z^2)}}\right)\right]dz$$

$$+\int_0^a\left[\frac{a^2z}{\sqrt{(a^2+z^2)}}\tan^{-1}\left(\frac{a}{\sqrt{(a^2+z^2)}}\right)-\tfrac{1}{2}az\log\left(\frac{2a^2+z^2}{a^2+z^2}\right)\right]dz$$

$$=\frac{a^3}{12}\left[28\sqrt{2}\tan^{-1}\left(\frac{1}{\sqrt{2}}\right)+8\log 2-24-7\pi-4\log 3\right]$$

$$+4a^3\int_0^{\pi/4}\sec^3\psi\tan^{-1}(\cos\psi)d\psi. \qquad 4a^3\int_0^{\pi/4}(\sec^3\psi\tan^{-1}(\cos\psi)d\psi$$

$$=4a^3\int_0^{\pi/4}(\sec^2\psi-\tfrac{1}{3}+\tfrac{1}{5}\cos^2\psi-\tfrac{1}{7}\cos^4\psi+\tfrac{1}{9}\cos^6\psi-\tfrac{1}{11}\cos^8\psi+\tfrac{1}{13}\cos^{10}\psi-\tfrac{1}{15}\cos^{12}\psi$$

$$+\ldots)d\psi=4a^3(1.02697-.06868\pi)=3.2448a^3.$$

$$\therefore N=1.5390a^3,\quad L=N/D=.5774a.$$

(a) $$L=\frac{2a\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\sin\theta\cos^2\phi\,d\theta\,d\phi}{\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\cos\phi\,d\theta\,d\phi}=\frac{4a}{\pi}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\sin\theta\cos^2\phi\,d\theta\,d\phi=a.$$

MISCELLANEOUS.

174. Proposed by L. E. DICKSON, Ph. D., Associate Professor of Mathematics, The University of Chicago.

By a linear transformation with integral coefficients modulo 2, reduce $f=\Sigma x_i^2+\Sigma x_ix_j$ $(i, j=1, \ldots, 2m;\ i<j)$ to a canonical form in which the variables are separated into pairs.

Solution by the PROPOSER.

Let $x_1=y_1+s$, $x_2=y_2+s$, where $s=\sum_{i=3}^{2m} x_i$. Then (mod 2),

$$f\equiv y_1y_2+y_1^2+y_2^2+F,\quad F=\Sigma\, x_ix_j\ (i, j=3, \ldots, 2m;\ i<j).$$

In F, set $x_3=y_3+t$, $x_4=y_4+t$, where $t=\sum_{i=5}^{2m} x_i$. Then

$$F\equiv y_3y_4+G,\quad G=\Sigma\, x_i^2+\Sigma\, x_ix_j\ (i, j=5, \ldots, 2m;\ i<j),$$

modulo 2. Since G is of the form f, repetitions of the process evidently lead to the required canonical form.

PROBLEMS FOR SOLUTION.

ALGEBRA.

317. Proposed by FRANCIS RUST, Allegheny, Pa.

Once, in classic days, Silenus lay asleep; a goat skin filled with wine near him. Dionysius passing by, profited, by siezing the skin, and drinking for two-thirds ($\frac{2}{3}$) of that time in which Silenus alone could have emptied said skin. At this point Silenus awoke, and seeing what was happening, snatched away the precious skin, and finished it.

Now, had both started together, and drank simultaneously, they would have consumed the wine skin in two hours less time. And, in this case, Dionysius' share would have been one-half as much as Silenus did secure, by waking and snatching the skin.

In what time would either one of them alone finish the goat-skin?

CALCULUS.

173. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

On one side of a circular pond a feet in radius is a duck. On the diametrically opposite side of the pond is a dog. Both begin to swim at the same time, the duck swimming around the circumference of the pond at the rate of m feet a minute, the dog swimming directly towards the duck at the rate of n feet per minute. How far will the dog swim in overtaking the duck?

174. Proposed by J. EDWARD SANDERS, Weather Bureau, Chicago, Ill.

About the vertices of a regular tetrahedron four spheres are drawn with radii equal to the edge of the tetrahedron. Find the volume common to them all.

175. Proposed by C. N. SCHMALL, 604 East 5th Street, New York City.

Explain fully why the circular measure of an angle is used in the calculus.

MECHANICS.

227. Proposed by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

Regarding the earth as a homogeneous sphere, radius R, acceleration at the surface g, investigate the motion of a sphere, radius b, moving through a straight tunnel between two points on the surface not diametrically opposite.

228. Proposed by J. E. ROSE, Mount Angel College, Mount Angel, Oregon.

AB, BC are two uniform rods freely hinged at B, whose weights are W, $4W$, and lengths $2a$, $4a$, respectively. The ends A, C of the rods are attached to small rings which slide on a rough horizontal wire. When the distance between the rings is the greatest for which equilibrium can exist, both of them are on the point of slipping. Find the coefficient of friction.

229. Proposed by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

Find the position of the center of pressure of a semi-elliptical area completely immersed in water, the area being vertical, the bounding axis major being inclined to the horizon at an angle β, and having one extremity in the surface of the water.

230. Proposed by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

A particle is projected from a distance $a=2r$ from the earth's center towards the earth with a velocity from infinity. If the earth is an airless homogeneous sphere, radius equal to the present mean radius and gravity as at present, with what velocity and in what time will it reach the center through an opening from surface to center?

BOOKS.

Plane and Spherical Trigonometry and Four-Place Tables of Logarithms. By William Anthony Granville, Ph. D., Sheffield Scientific School, Yale University. 8vo. Cloth. xi+264 pp. +38 pp. of Tables. Price, $1.25. Boston and Chicago: Ginn & Co.

The subject of Trigonometry as presented in this book is in accordance with the latest and most approved methods. Trigonometric Functions are defined as ratios, first for angles less than a right angle and then extended to angles in general. A valuable feature of the book is the fact that the degree of accuracy which may be expected in a result found by the aid of the tables is clearly indicated. In the treatment of oblique spherical triangles the author makes use of the Principle of Duality, whereby nearly half of the work usually required in deriving the standard formulae is saved and the usual six cases are reduced to three. A large number of illustrative problems are worked out under each topic and a large number of well graded exercises are given. The typography and mechanical execution of the book are very beautiful and artistic. B. F. F.

An Introduction to the Study of Integral Equations. By Maxime Bôcher, B. A., Ph. D., Professor of Mathematics in Harvard University. 8vo. Paper cover. 3+72 pages. Price, 2s, 6d. Cambridge: Cambridge University Press.

In this tract, the author has presented the main portions of the Theory of Integral Equations in a readable and accurate form. It has followed roughly the line of historical development, beginning with Abel's Mechanical Problem. It is believed that the careful student is here furnished with a firm foundation upon which he may build further study and investigation, while the more superficial reader will be satisfied with the concise and precise statements of results. B. F. F.

Spezielle Ebene Kurven, von Dr. Heinrich Wieleitner, Gymnasiallehrer am hum. Gymnasium Speyer, mit 189 Figuren im Text. Gr. 8°. xvi+409 Seiten. Preis, in Leinwand gebunden, M. 12. Leipzig: G. J. Goeschen'sche Verlogshandlung.

This is a very valuable book on special algebraic and trancendental plane curves. Here is to be found a full treatment of the Cissoid, Conchoid, Lemniscate, Cartesian Ovals, Roulettes, Spirals, etc., etc. Many generalizations of these curves are made and their properties are derived. The book is of special value to the student of modern Geometry. B. F. F.

Correlation of Efficiency in Mathematics and Efficiency in Other Subjects. A Statistical Study. By Professor H. L. Rietz, Ph. D., and Miss Imogene Shade, A. B., November, 1908. 20 pages. Price, 35 cents.

This is the latest number of *The University Studies*, published by the University of Illinois. It is a scientific comparison of the grade of work students do in mathematics, foreign languages and natural sciences. The data, covering a period of nineteen years, were procured from the Registrar of the University of Illinois. The method of investigation may be characterized as the statistical method of Galton and Pearson. The paper presents some new points in the theory of statistics, but the main result of general interest is the discovery that a student who is good in mathematics is also good in foreign languages or in natural sciences, or *vice versa*. The authors call attention to the important educational value of these results. L. J. P.

First Course in Calculus. By E. J. Townsend, Ph. D., (Goettingen) Professor of Mathematics, University of Illinois, and G. A. Goodenough, M. E. (Illinois), Associate Professor of Mechanical Engineering, University of Illinois. Large 8vo. Cloth. xii+466 pages. Price, $2.00. New York: Henry Holt & Co.

Among the excellent works on the Calculus which have recently appeared, the one under consideration must be accorded a very high rank. The presentation of the fundamental principles of the Calculus are rigorously established, the arrangement of subjects is logical, and the material selected eminently practical. The illustrations and applications from Physics are commendable. The book is one well suited for class room purposes and also for self-instruction and supplementary reading. This work is a worthy contribution to the literature of the subject. B. F. F.

The Principles of Mechanics for Students of Physics and Engineering. By Henry Crew, Ph. D., Fayerweather Professor of Physics in Northwestern University. 8vo. Cloth. ix+295 pp. Price, $1.50. New York: Longmans, Green & Co.

The author's efforts, as stated in the preface, have been: (1) To lead the student to clear dynamical views in the shortest possible time, without sacrificing him on the altar of logic, yet pursuing a route which he can afterwards follow with safety; (2) to build the discussion upon a few simple experiments and upon definitions which convey at once the physical meaning of the quantities defined; (3) to follow the example of Föppl in using vector analysis merely to present a clear, simple, and accurate picture of the facts, reserving the Cartesian analysis for purposes of computation; (4) to confine the treatment to that part of mechanics which is common ground for the physicist and the engineer; (5) to reduce the inherent difficulties of the subject to a minimum by treating dynamics in two analogous parts—rotational and translational—such that if either one is given the other may be immediately deduced; and (6) to employ only two systems of units, the absolute C. G. S. and the "British Engineers." In our opinion, the author has executed his purpose most admirably and has given the teacher of mechanics a teachable book.

B. F. F.

A Manual of Practical Physics for Students of Engineering. By Ervin Sidney Ferry, Professor of Physics, Purdue University, and Arthur Taber Jones, Assistant Professor of Physics, Purdue University. Vol. I. Fundamental Measurements and Properties of Matter. Heat. 8vo. Cloth. xi+273 pp. Price, $1.75. New York: Longmans, Green & Co.

This work furnishes the student of pure and applied science with a manual of the theory and manipulation of those measurements which bear most directly upon his work in other departments of study and also upon his professional career. The description and theory of each experiment is very clear and the material well selected. However, the book, in our judgment, would have been improved had the authors inserted a blank form for the data and results of each experiment. It is the purpose of the authors to complete the course in three volumes, the last two of which are in preparation. B. F. F.

Théorie et Applications des Equations du Second Degré a l'usage des Eléves de Seconde et Primiere C et D et de Mathematiques A et B. Par J. Juhel-Rénoy. Paris: Vubert et Nony Editeurs. 251 pages.

This book treats very fully the equation of the second degree, also rational fractions whose numerators and denominators are of the second degree, and inequalities of the second degree. Biquadratic equations reducible to quadratics are discussed in Book III, chapter I and reciprocal equations in chapter II of this book and irrational equations in the last chapter, III. B. F. F.

5

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PERIODIC DECIMAL FRACTIONS.

By ELIZABETH R. BENNETT, The University of Illinois.

Although decimal fractions present themselves very early in the mathematical course yet they offer many questions to which it is difficult to find an answer in the text-books or even in the mathematical encyclopedias. Some of these questions are of such an elementary type that their answers can be appreciated even by students of little mathematical training. Moreover, periodic decimals offer "an unlimited amount of material for practice, leading to results whose beauty and easy verification have a great charm."*

The main object of the present paper is to collect the principal theorems relating to periodic decimals and to present them in a form which can easily be understood. It is believed that a useful service may thus be rendered to teachers of secondary mathematics, especially since these theorems are so scattered and relate to such an elementary subject. In addition to this, the writer has proved some of the results by more modern methods, especially by using elementary properties of group theory. These methods of proof are the only elements of novelty claimed for the present article. An outline of these methods was published by Professor Miller, *Bulletin of the American Mathematical Society*, Vol. XIV, 1908, page 356.

In the *Nouvelles Annales de Mathématiques* for 1842 M. Catalan has summarized the principal theorems then known regarding decimals, basing the greater number of his proofs on Fermat's theorem. Another summary by Laisant and Beaujeux which includes several additional theorems appears in the same magazine for 1868. The proofs given in this article are developed from the standpoint of radix fractions, *i. e.*, fractions developed according to the powers of a base. Since only a few of the theorems stated in these two summaries, or elsewhere, are found in the mathematical encyclopaedias or in the texts on arithmetic, algebra, or number theory, it has seemed desirable to present briefly the more important ones in this article. Although no attempt has been made to exhibit new theorems, in several instances, the proofs have been modernized by the application of elementary group theory principles.

*Weber und Wellstein, *Encyklopaedie der Elementar-Mathematik*, Vol. 1, 1906, p. 258.